Symbolic Logic 3: Metalogic 1

We have so far learnt a language for translating some simple forms of argument and evaluating them for deductive validity.

An argument is a sequence of sentences, regimented so the premises come first followed by 'therefore' and then the conclusion. We have tested them by using trees in which we start by forming the set of the premises and add the negation of the conclusion. If that set is inconsistent we say that the original argument is valid; if the set is not inconsistent (if a branch remains open) then we say that the argument exhibits a truth-functionally invalid structure.

Metalogic

Most of the time we have simply used the artificial language to construct truth-table and trees. But it is also possible to study how the language itself works, what can and cannot be done with it. This is a matter of looking at the language from the outside, at a higher level. It is often called meta-logic for that reason.

We have in effect two effect two perspectives on being a valid argument. One, the semantic view, says that in a valid deductive argument you can't have all the premises true and the conclusion false. We can describe such a situation formally by writing, in our metalanguage:

$$\Gamma \models \phi$$

The other, a syntactic account, is a matter of saying that in constructing a tree all the branches close (i.e. there is a contradiction on all of them). We can write that:

$$\Gamma \vdash \phi$$

For truth-functional logic, one of the most important metalogical theses is that these two characterisations of entailment coincide.

These signs are often called turnstiles.

While we have been concentrating on situations in which we have premises and a conclusion, we can generalise these notions to cases where there are no premises or conclusion.

So we can have situations in which

$$\Gamma \models$$

That is a situation in which the formulae in Γ cannot all be true, i.e. Γ is inconsistent.

We can also have situations in which

$$\models \phi$$

i.e. the formula for which ϕ stands cannot be false - it is a truth of logic or a tautology.

As Bostock says, $\models$ says in effect the formulae on the left-side cannot all be true while those on the right are false.

There are several facts about entailment that can be rigorously proved. For our purposes it is enough simply to grasp what they are saying.

Extension theorem (monotonicity)

If Γ and Δ are finite sets of formulae, possibly empty, and ψ is a formula, then:

$$\text{if } \Gamma \models \phi \text{ then } \Gamma, \Delta \models \phi$$

Can't make a valid argument invalid by adding premises.

Repetition theorem

If Γ is a set of formulae and ϕ is a formula in Γ, then $\Gamma \models \phi$.

Cut

If Γ is a finite set of formulae and ϕ and ψ are formulae, then

$$\text{if } \Gamma \models \phi \text{ and } \Gamma, \phi \models \psi, \text{ then } \Gamma \models \psi$$

Assume $\Gamma \models \phi$ and $\Gamma, \phi \models \psi$, and let A be a structure in which ψ is defined and all the formulae of Γ are true. We need to show that ψ is true in A. ϕ may not be defined in A but we can always add truth-value assignments to A so as to get a structure B in which ϕ is defined. Since $\Gamma \models \phi$, ϕ is true in B. Therefore, since $\Gamma, \phi \models \psi$, the formula ψ must also be true in B. But then ψ is true in A too, since B is only A with pieces added. (Hodges, pp. 132-3)

Transitivity of semantic entailment

If ϕ, ψ and χ are formulae, then

$$\text{if } \phi \models \psi \text{ and } \psi \models \chi, \text{ then } \phi \models \chi.$$

Prove by Extension followed by Cut.

Substitution theorem

Every substitution instance of a correct semantic sequent is again correct.

Logical equivalence holds between ϕ and ψ if $\phi \models \psi$ and $\psi \models \phi$. Logical equivalence is reflexive ($\phi \models \phi$); symmetrical (if ϕ is logically equivalent to ψ then ψ is logically equivalent to ϕ) and transitive (if ϕ is logically equivalent to ψ and ψ is logically equivalent to χ then ϕ is logically equivalent to χ).

Congruence of Logical Equivalence

If, in a correct semantic sequent, every formula is replaced by a formula logically equivalent to it, then the resulting sequent is also correct.

Equivalence theorem

Replacing logically equivalent sub-formulae results in a formula logically equivalent to what you start from.

This allows one to convert formulae into equivalent formulae using only negation and disjunction or negation and conjunction. When these are of certain forms they are known as *normal* forms.

Interpolation theorem

If ϕ and ψ are formulae such that $\phi \models \psi$, and at least one sentence letter occurs in both ϕ and ψ, then there is a formula such that

$$\phi \models \chi, \chi \models \psi$$

and every sentence letter in χ is in both ϕ and ψ. (χ is known as an *interpolant* for the sequent '$\phi \models \psi$'.)

Syntactic entailment

If all branches of a tableau close then the set of statements is inconsistent:

$$\Gamma \vdash.$$

If there is a closed tableau generated by Γ than every tableau generated by Γ can be extended to a closed one.

Completeness

A syntactic sequent is correct if and only if the corresponding semantic sequent is correct. This amounts to showing that a finite set of formulae is syntactically inconsistent precisely if it is semantically inconsistent.